

On covariant expansion of the gravitational Stückelberg trick

Xian Gao^{1,*}

¹*Department of Physics, Tokyo Institute of Technology,
2-12-1 Ookayama, Meguro, Tokyo 152-8551, Japan
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A new approach to expanding the “Stückelbergized” fiducial metric in a covariant manner is developed. The idea is to consider the curved 4-dimensional space as a codimension-one hypersurface embedded in a 5-dimensional Minkowski bulk, in which the 5-dimensional Goldstone modes can be defined as usual. After solving one constraint among the five 5-dimensional Goldstone modes and projecting onto the 4-dimensional hypersurface, we are able to express the “Stückelbergized” fiducial metric in terms of the 4-dimensional Goldstone modes as well as 4-dimensional curvature quantities. We also compared the results with expressions got using the Riemann Normal Coordinates (RNC) in Gao et al [Phys. Rev. D90, 124073 (2014)] and find that, after a simple field redefinition, results got in two approaches exactly coincide.

I. INTRODUCTION

Attempts to explain the primordial and late time accelerating expansion of our Universe stimulate the study of theories beyond general relativity (GR) with a cosmological constant (see [1, 2] for reviews and [3] for a short introduction). Such theories typically contain additional degrees of freedom to the two tensor modes of GR. Instead of introducing extra fields by hand, one approach to these new degrees of freedom is to construct effective theories with less gauge redundancies comparing with GR. This can be achieved most straightforwardly by introducing a fiducial metric $\bar{f}_{\mu\nu}$, which does not change under the coordinates transformation and thus explicitly breaks the general covariance.

If $\bar{f}_{\mu\nu}$ is degenerate and has only one nonvanishing time-like eigenvector, one gets the effective field theory (EFT) of inflation [4, 5] and recent so-called “theories beyond Horndeski” [6, 7], where the time diffeomorphism is broken and thus generally an additional scalar mode arises [8–10]. Spatial symmetries may be broken by considering $\bar{f}_{\mu\nu}$ with nonvanishing spacelike eigenvectors. If the number of nonvanishing spacelike eigenvectors is equal to the spatial dimension, one arrives at some sort of massive gravity theories (typically Lorentz-violating, e.g. [11, 12]). A nondegenerate $\bar{f}_{\mu\nu}$ breaks all spacetime symmetries, through which a Lorentz-invariant massive gravity can be constructed [13] (see [14, 15] for reviews). In this note, we concentrate on the case of a nondegenerate $\bar{f}_{\mu\nu}$.

The fiducial metric $\bar{f}_{\mu\nu}$, whose existence breaks the general covariance explicitly, can always be thought of as the “gauge-fixed” version of some covariant tensor field. This is just the idea of gravitational Stückelberg trick, which dates from [16, 17] in the study of open string theory. We may promote the fiducial metric $\bar{f}_{\mu\nu}$ as [17, 18]

$$\bar{f}_{\mu\nu} \rightarrow f_{\mu\nu}(x) \equiv \bar{f}_{ab}(\phi(x)) \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu}, \quad (1)$$

where the “Stückelbergized” fiducial metric $f_{\mu\nu}$ transforms as a tensor as long as each of the four (we are working with 4-

dimensional spacetime) Stückelberg fields $\{\phi^a\}$ transform as scalars under a general coordinates transformation of spacetime. The fixed $\bar{f}_{\mu\nu}$ is recovered by choosing the so-called “unitary gauge” with $\phi^a \rightarrow \delta_\mu^a x^\mu$. In practice, we may expand the Stückelberg fields around the unitary gauge and concentrate on the behavior of their fluctuations [18]

$$\phi^a - \delta_\mu^a x^\mu \equiv -\delta_\mu^a \hat{\pi}^\mu. \quad (2)$$

When the fiducial metric $\bar{f}_{\mu\nu}$ is flat, it has been well-known that in the so-called decoupling limit (some limit of energy scales where the interactions among different types of degrees of freedom get simplified), $\hat{\pi}_\mu$ defined in (2) behaves as a spacetime vector. In this case, we can fix a gauge in which the helicity-1 and helicity-0 parts of the graviton are encoded in $\hat{\pi}_\mu$ [19]. It is just in this way that the Boulware-Deser ghost [20] can be seen most transparently [21, 22]. This argument, however, cannot be simply applied to a general fiducial metric $\bar{f}_{\mu\nu}$. First, naively plugging (2) into (1) would inevitably yield noncovariant expressions [23, 24]. More seriously, as was well explained in [23, 25, 26], $\hat{\pi}_\mu$ defined in (2) is not a vector and does not capture the helicity-1 and helicity-0 modes of the graviton correctly, either when going beyond the decoupling limit or when the fiducial metric has curvature.

This problem was systematically solved in [24] by employing the the Riemann Normal Coordinates (RNC), where a covariant formulation of the Stückelberg expansion with a general fiducial metric was developed. A decoupling limit analysis similar to the case of a flat fiducial metric was consistently performed in [24], where the helicity modes can be characterized correctly¹. On the other hand, when dealing with the de Sitter fiducial metric, an alternative approach to the covariant Stückelberg expansion was developed in [23]. The idea is to embed the d -dimensional de Sitter space into a $(d+1)$ -dimensional Minkowski background, in which the Goldstone modes can be identified as in (2). Then by projecting to the d -dimensional de Sitter space, (1) can be expanded in terms of the correct helicity modes in d dimensions, in a covariant man-

* Email: gao@th.phys.titech.ac.jp

¹ See also [27–30] for related progresses on the Stückelberg analysis and decoupling limit of massive gravity around a general background.

ner. The purpose of this note, is to develop this technique further, and more systematically, by considering a general fiducial metric.

This note is organized as follows. In Sec.II, we briefly review the main results in [24] on the covariant Stückelberg expansion based on RNC. In Sec.III we establish the basic formalism of embedding the 4-dimensional curved space into a 5-dimensional Minkowski one, and determine the “covariant” Goldstone modes in the 4 dimensions. In Sec.IV, we derive the covariant expansions for the fiducial metric in terms of 4-dimensional quantities, and compare them with the corresponding results in [24]. Finally we briefly summarize in Sec.V.

II. COVARIANT EXPANSION BASED ON THE RIEMANN NORMAL COORDINATES

In [24], the Riemann Normal Coordinates (RNC) was employed to derive covariant expressions for the Stückelberg expansion in the presence of a general fiducial metric². The idea is to consider a one-parameter family of diffeomorphisms of the spacetime with parameter λ :

$$\phi_\lambda : p \mapsto \phi_\lambda(p), \quad (3)$$

where p is a given spacetime point. The Stückelberg fields at point p are defined as the coordinate values of its image $\phi_\lambda(p)$ with $\lambda = -1$:

$$\phi^\mu|_p \equiv x^\mu|_{\phi_{-1}(p)}. \quad (4)$$

The “covariant” Goldstone modes π^μ are thus defined as the standard RNC’s of the image point $\phi_{-1}(p)$, i.e., the tangent vector of the geodesic at point p connecting p and its image $\phi_{-1}(p)$:

$$\begin{aligned} \phi^\mu|_p &= x_0^\mu - \pi^\mu - \frac{1}{2}\bar{\Gamma}_{\nu\rho}^\mu \pi^\nu \pi^\rho \\ &+ \frac{1}{6}(\partial_\nu \bar{\Gamma}_{\rho\sigma}^\mu - 2\bar{\Gamma}_{\nu\lambda}^\mu \bar{\Gamma}_{\rho\sigma}^\lambda) \pi^\nu \pi^\rho \pi^\sigma + \dots, \end{aligned} \quad (5)$$

where $x_0^\mu \equiv x^\mu|_p$, and $\bar{\Gamma}_{\nu\rho}^\mu$ is the Christoffel symbol associated with $\bar{f}_{\mu\nu}$. Comparing with (2), (5) also implies a nonlinear relation between $\hat{\pi}_\mu$ and π_μ .

By plugging (5) into (1) and carefully dealing with the Christoffel symbols and their derivatives, it is possible to expand the “Stückelbergized” fiducial metric $f_{\mu\nu}$ covariantly in terms of π_μ , $\bar{R}_{\mu\nu\rho\sigma}$ as well as their covariant derivatives (with respect to $\bar{f}_{\mu\nu}$). An equivalent and more convenient approach, is to evaluate (see [24] for details)

$$\begin{aligned} f_{\mu\nu} &= e^{-\mathcal{L}\pi} \bar{f}_{\mu\nu} \equiv \sum_{n=0} \frac{(-1)^n}{n!} \mathcal{L}_\pi^n \bar{f}_{\mu\nu} \\ &= \sum_{n=0} f_{\mu\nu}^{(n)\text{RNC}}, \end{aligned} \quad (6)$$

where $\mathcal{L}_\pi^n \bar{f}_{\mu\nu}$ is the n -th order Lie derivative of $\bar{f}_{\mu\nu}$ with respect to π^μ . Straightforward calculations yield³

$$f_{\mu\nu}^{(1)\text{RNC}} = -2\bar{\nabla}_{(\mu}\pi_{\nu)}, \quad (7)$$

$$f_{\mu\nu}^{(2)\text{RNC}} = \bar{\nabla}_\mu \pi_\rho \bar{\nabla}_\nu \pi^\rho - \bar{R}_{\mu\rho\nu\sigma} \pi^\rho \pi^\sigma, \quad (8)$$

$$f_{\mu\nu}^{(3)\text{RNC}} = \frac{1}{3} \pi^\alpha \pi^\beta (\pi^\rho \bar{\nabla}_\rho \bar{R}_{\mu\alpha\nu\beta} + 4\bar{\nabla}_{(\mu} \pi^\rho \bar{R}_{\nu)\alpha\rho\beta}), \quad (9)$$

$$\begin{aligned} f_{\mu\nu}^{(4)\text{RNC}} &= -\frac{1}{12} (\bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{R}_{\mu\rho\nu\sigma} - 4\bar{R}_{\mu\alpha\beta\lambda} \bar{R}_{\nu\rho\sigma}{}^\lambda) \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma \\ &+ \frac{1}{2} \bar{\nabla}_\rho \bar{R}_{\alpha\beta\sigma(\mu} \bar{\nabla}_{\nu)} \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma \\ &- \frac{1}{3} \bar{R}_{\rho\alpha\sigma\beta} \pi^\alpha \pi^\beta \bar{\nabla}_\mu \pi^\rho \bar{\nabla}_\nu \pi^\sigma. \end{aligned} \quad (10)$$

Please note in deriving (8)-(10), $\pi^\nu \bar{\nabla}_\nu \pi^\mu = 0$ is used, since π^μ is the tangent vector of geodesics.

III. STÜCKELBERG BY EMBEDDING

A different approach to the covariant Stückelberg expansion was introduced in [23] in the study of massive gravity on de Sitter background. This approach is based on the observation that the d -dimensional de Sitter space can be embedded into a $(d+1)$ -dimensional Minkowski one, in which the Goldstone modes can be identified easily as in (2) and the Stückelberg expansion of the fiducial metric can be performed as usual. Then the d -dimensional quantities can be got, in a covariant manner, by simply projecting the $(d+1)$ -dimensional ones onto the d -dimensional de Sitter space.

In general, to embed an arbitrary d -dimensional space into a $(d+1)$ -dimensional Minkowski one is not always possible. While as the first attempt, in this note, we restrict ourselves to the subclass of 4 dimensional metrics which can be embedded (at least locally) into a 5 dimensional flat bulk. In this case, the corresponding Stückelbergized fiducial metric is given by

$$f_{\mu\nu} = \eta_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu}, \quad (11)$$

where $\{X^M\}$ with $M = 0, 1, 2, 3, 4$ are Cartesian coordinates of the 5-dimensional Minkowski space. As we shall see, in the 5-dimensional flat bulk, the unitary gauge and the decomposition of the Stückelberg fields can be performed in a standard manner. All the subtleties are thus in the projection from 5 dimensions to 4 dimensions.

It is convenient to introduce another set of coordinates $\{x^a\}$ for the 5-dimensional Minkowski bulk such that its metric takes the form

$$ds^2 = \eta_{ab} dx^a dx^b, \quad (12)$$

with

$$\eta_{ab}(x) = \eta_{MN} \frac{\partial X^M(x)}{\partial x^a} \frac{\partial X^N(x)}{\partial x^b}. \quad (13)$$

² This is in fact a standard approach in order to define perturbations “covariantly”, as in the well-known background field method [31]. For massive gravity, RNC was also suggested in [25] (footnote 5).

³ Note in [24], $f_{\mu\nu}^{(n)\text{RNC}}$ were evaluated only up to the cubic order in π^μ . Here we also evaluate $f_{\mu\nu}^{(4)\text{RNC}}$ for late convenience.

Note we also have $\eta^{ab} \frac{\partial X^M(x)}{\partial x^a} \frac{\partial X^N(x)}{\partial x^b} = \eta^{MN}$.

The embedding of the 4-dimensional hypersurface into the 5-dimensional bulk can be parametrized by a constraint among coordinates $\{X^M\}$:

$$\Phi(X^M) = 0, \quad (14)$$

which is a scalar function under diffeomorphism and global Lorentz transformation of $\{X^M\}$. Since the hypersurface is codimensional one, its normal vector (with normalization $\eta^{MN} n_M n_N = 1$) is thus given by

$$n_M \equiv N \partial_M \Phi, \quad (15)$$

with $\partial_M \equiv \partial \Phi / \partial X^M$ and

$$N = \frac{1}{\sqrt{\eta^{MN} \partial_M \Phi \partial_N \Phi}}. \quad (16)$$

Here the sign of n_M is chosen to be compatible with the fact that n_M is spacelike, such that the induced metric on the 4 dimensional hypersurface

$$h_{MN} = \eta_{MN} - n_M n_N, \quad (17)$$

is Lorentzian.

A. Unitary gauge

Up to now, the formalism is quite general. We can now fix a gauge (unitary gauge) by choosing a specified mapping between the two set of coordinates:

$$x^a \mapsto X^M \equiv \bar{X}^M(x), \quad (18)$$

which corresponds to a special choice of $\{x^a\}$ -coordinates adapted to the embedding, i.e.,

$$\{x^a\} = \{x^\mu, y\}, \quad \text{with } \mu = 0, 1, 2, 3, \quad (19)$$

such that the metric of 5-dimensional bulk in this particular $\{x^a\}$ -coordinates takes the form

$$\begin{aligned} ds^2 &= \bar{h}_{ab} dx^a dx^b \\ &= \bar{f}_{\mu\nu} dx^\mu dx^\nu + 2N_\mu dx^\mu dy + (N^2 + N_\mu N^\mu) dy^2, \end{aligned} \quad (20)$$

where $\bar{f}_{\mu\nu}$ is just the fixed induced metric on the hypersurface, which we treat as being given beforehand. In (20) we write \bar{h}_{ab} in order to emphasize it is the expression in the unitary gauge. It is always possible to choose y -coordinate to be normal to the hypersurface so that $\bar{h}_{\mu y} = N_\mu = 0$. Here we keep $N_\mu \neq 0$ for generality, while as we shall see later, all the contributions from N_μ drop out in the final expressions. At this point, keep in mind that N and N_μ must be determined by $\bar{f}_{\mu\nu}$ in order to make sure that \bar{h}_{ab} is indeed describing a flat space.

In the following, we use

$$e_a^M \equiv \frac{\partial \bar{X}^M(x)}{\partial x^a}, \quad (21)$$

for short. In this unitary gauge, the normal vector to the hypersurface is given by, in $\{X^M\}$ -coordinate:

$$n^M = \frac{1}{N} (e_y^M - N^M), \quad (22)$$

and in $\{x^a\}$ -coordinates:

$$n_a \equiv \{0, \dots, 0, n_y\} = \{0, \dots, 0, N\}. \quad (23)$$

The induced metric on the hypersurface (17) thus becomes

$$h_{ab} = \eta_{ab} - n_a n_b. \quad (24)$$

B. Goldstone modes on the hypersurface

Deviation from the unitary gauge can be achieved by choosing a different mapping $x^a \mapsto X^M$, which is parametrized by

$$X^M(x) = \bar{X}^M(x) - \pi^M(x), \quad (25)$$

where $\pi^M(x)$ are five Goldstone modes in the 5-dimensional bulk. Note (25) is simply a copy of (2). Now comes the crucial point. Since the 4-dimensional hypersurface is treated as being fixed and thus nondynamical, all gauge choices should satisfy the constraint (14). That is, we must have

$$\Phi(\bar{X}^M(x) - \pi^M(x)) = 0, \quad (26)$$

which eliminates one degree of freedom among the five Goldstone modes π^M . Thus we are left with only four gauge modes, as is expected. This is different from the case where the hypersurface is dynamical (e.g. galileons from induced metric [32–36]), where the position of the hypersurface becomes a dynamical variable, which cannot be gauged away.

The components of π_M in $\{x^a\}$ -coordinate read

$$\pi_a \equiv \eta_{MN} e_a^M \pi^N = \{\pi_\mu, \pi_y\}. \quad (27)$$

Equivalently, we may write

$$\pi^M \equiv e_a^M \pi^a = e_\mu^M \pi^\mu + e_y^M \pi^y. \quad (28)$$

For later convenience, we also define

$$\pi^\perp \equiv n_M \pi^M = n_a \pi^a = n_y \pi^y \equiv N \pi^y. \quad (29)$$

From the 4-dimensional point of view, it may be convenient to choose the four Goldstone modes to be π_μ , i.e., the direct projection of π^M on the hypersurface. The crucial point is, π^\perp must be determined by π^μ through the constraint (26) as

$$\pi^\perp = \pi^\perp(\pi^\mu), \quad (30)$$

which is a scalar under $\{x^\mu\}$ reparametrization. The embedding of the 4-dimensional hypersurface as well as the definition of 5-dimensional and 4-dimensional Goldstone modes are illustrated in Fig.1.

In general, we may solve π^\perp in terms of π^μ perturbatively. To this end, we expand (26) around $\Phi(\bar{X}) = 0$, which yields

$$\sum_{n=1} (-1)^n \frac{\partial^n \Phi}{\partial \bar{X}^{M_1} \dots \partial \bar{X}^{M_n}} \pi^{M_1} \dots \pi^{M_n} = 0. \quad (31)$$

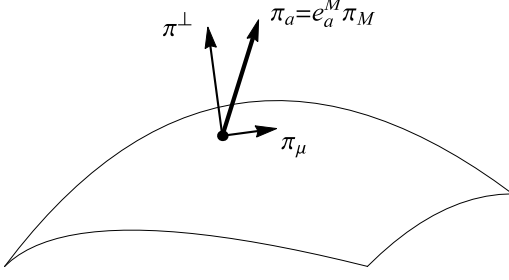


FIG. 1. Illustration of the embedding of the 4-dimensional hypersurface into the 5-dimensional Minkowski background and the definition of Goldstone modes.

In $\{x^a\}$ -coordinate, the constraint (31) becomes

$$\sum_{n=1} (-1)^n \Phi_{;a_1 \dots a_n} \pi^{a_1} \dots \pi^{a_n} = 0, \quad (32)$$

where

$$\Phi_{;a_1 \dots a_n} \equiv e_{a_1}^{M_1} \dots e_{a_n}^{M_n} \frac{\partial^n \Phi}{\partial \bar{X}^{M_1} \dots \partial \bar{X}^{M_n}} \equiv \nabla_{a_1} \dots \nabla_{a_n} \Phi. \quad (33)$$

Note $\Phi_{;a_1 \dots a_n}$ is symmetric since the 5-dimensional bulk is essentially flat.

By definition,

$$\nabla_a \Phi = \frac{1}{N} n_a, \quad (34)$$

and a further derivative yields

$$\nabla_a \nabla_b \Phi = -\frac{1}{N} (n_a n_b \rho - 2n_{(a} a_{b)} - K_{ab}), \quad (35)$$

with shorthands

$$\rho \equiv \mathcal{L}_n \ln N, \quad a_a \equiv -D_a \ln N. \quad (36)$$

In the above, D_a is the tangent covariant derivative (compatible with h_{ab} in (24)), $K_{ab} \equiv D_b n_a$ is the extrinsic curvature. The third derivative is decomposed to be⁴ [7, 10]

$$\nabla_a \nabla_b \nabla_c \Phi = \frac{1}{N} \left(n_a n_b n_c \omega + 2n_a n_{(b} \lambda_{c)} + n_a \zeta_{bc} + \lambda_a n_b n_c + 2\xi_{a(b} n_{c)} + \chi_{abc} \right), \quad (37)$$

with

$$\omega \equiv \rho^2 - \mathcal{L}_n \rho - 2a_a a^a, \quad (38)$$

$$\lambda_a \equiv \mathcal{L}_n a_a - 2\rho a_a - 2K_a^{c'} a_{c'}, \quad (39)$$

$$\zeta_{ab} \equiv 2a_a a_b - \rho K_{ab} + \mathcal{L}_n K_{ab} - 2K_{ac'} K_b^{c'}, \quad (40)$$

$$\xi_{ab} \equiv a_a a_b + D_a a_b - \rho K_{ab} - K_a^{b'} K_{b'}^b, \quad (41)$$

$$\chi_{abc} \equiv a_a K_{bc} + 2K_{a(b} a_{c)} + D_a K_{bc}, \quad (42)$$

⁴ Here (37) and (38)-(42) are slightly different from those in [7, 10] since now the normal to the hypersurface is spacelike.

where ρ and a_a are defined in (36). The full decomposition of $\nabla_a \nabla_b \nabla_c \nabla_d \Phi$ is rather involved and we prefer not to present it in this note. For our purpose to solve π^\perp up to the fourth order in π_μ , only the purely tangent part of $\nabla_a \nabla_b \nabla_c \nabla_d \Phi$ is needed, which reads

$$\nabla_a \nabla_b \nabla_c \nabla_d \Phi \supset \frac{1}{N} \chi_{abcd}, \quad (43)$$

with [37]

$$\chi_{abcd} = K_{ab} \zeta_{cd} + K_{ac} \xi_{bd} + K_{ad} \xi_{bc} + a_a \chi_{bcd} + D_a \chi_{bcd}, \quad (44)$$

where ζ_{ab} etc are defined in (40)-(42).

Supposing that π^\perp can be solved in terms of π_μ perturbatively as

$$\pi^\perp = \sum_{n=1} \pi_{(n)}^\perp, \quad (45)$$

where $\pi_{(n)}^\perp \sim \mathcal{O}((\pi_\mu)^n)$. Plugging (34), (35) and (37) into (32) and using the definition for π^\perp and π_μ , after some manipulations we have

$$\begin{aligned} \pi^\perp = & \frac{1}{2} \pi^\mu \pi^\nu \bar{K}_{\mu\nu} - \frac{1}{6} \pi^\mu \pi^\nu \pi^\rho \bar{\nabla}_\mu \bar{K}_{\nu\rho} \\ & + \frac{1}{24} \pi^\mu \pi^\nu \pi^\rho \pi^\sigma (3\bar{K}_{\mu\nu} \bar{K}_{\rho\lambda} \bar{K}_\sigma^\lambda + \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{K}_{\rho\sigma}) \\ & + \mathcal{O}((\pi_\mu)^5). \end{aligned} \quad (46)$$

(46) explicitly depends on the extrinsic curvature $\bar{K}_{\mu\nu}$ and its derivatives. While from the 4-dimensional point of view, the fiducial metric should not “know” anything about the embedding. As we shall see in the next section, in the final expressions for the Stückelbergized fiducial metric $f_{\mu\nu}$ (11), all the dependence on the extrinsic curvature gets suppressed after using the Gauss relation and a simple field redefinition.

IV. COVARIANT EXPANSION OF THE FIDUCIAL METRIC

We are now ready to expand the “Stückelbergized” fiducial metric given by

$$f_{\mu\nu} = \mathfrak{h}_{\mu\nu} \quad (47)$$

with \mathfrak{h}_{ab} given in (13), in terms of π_μ as well as 4-dimensional curvature quantities. Expanding around the unitary gauge by plugging (25) into (13), we get

$$f_{\mu\nu} = \bar{f}_{\mu\nu} - 2\eta_{MN} \partial_{(\mu} \bar{X}^M \partial_{\nu)} \pi^N + \eta_{MN} \partial_\mu \pi^M \partial_\nu \pi^N, \quad (48)$$

which is exact. Note in deriving (48), we used $\eta_{MN} \partial_\mu \bar{X}^M \partial_\nu \bar{X}^N \equiv \bar{f}_{\mu\nu}$. Our purpose is to rewrite (48) in a covariant manner in terms of 4 dimensional quantities.

For the second term in (48), it is easy to show that (see (A3))

$$\begin{aligned} \eta_{MN} \partial_\mu \bar{X}^M \partial_\nu \pi^N &= e_\mu^M e_\nu^N \partial_N \pi_M \\ &\equiv \bar{\nabla}_\nu \pi_\mu + \bar{K}_{\mu\nu} \pi^\perp, \end{aligned} \quad (49)$$

where π^\perp is defined in (29). For the last term in (48), first we have

$$\begin{aligned}
& \eta_{MN} \partial_\mu \pi^M \partial_\nu \pi^N \\
&= (\bar{h}^{ab} e_a^M e_b^N) e_\mu^{M'} e_\nu^{N'} \partial_{M'} \pi_M \partial_{N'} \pi_N \\
&= \bar{h}^{\rho\sigma} (e_\rho^M e_\mu^{M'} \partial_{M'} \pi_M) (e_\sigma^N e_\nu^{N'} \partial_{N'} \pi_N) \\
&\quad + \bar{h}^{\rho y} (e_\rho^M e_\mu^{M'} \partial_{M'} \pi_M) (e_y^N e_\nu^{N'} \partial_{N'} \pi_N) \\
&\quad + \bar{h}^{y\sigma} (e_y^M e_\mu^{M'} \partial_{M'} \pi_M) (e_\sigma^N e_\nu^{N'} \partial_{N'} \pi_N) \\
&\quad + \bar{h}^{yy} (e_y^M e_\mu^{M'} \partial_{M'} \pi_M) (e_y^N e_\nu^{N'} \partial_{N'} \pi_N), \quad (50)
\end{aligned}$$

where $\bar{h}^{\mu\nu}$ etc are the components of the matrix inverse of \bar{h}_{ab} in (20). While using (22), we have

$$\begin{aligned}
& e_y^M e_\mu^{M'} \partial_{M'} \pi_M \\
&= N n^M e_\mu^{M'} \partial_{M'} \pi_M + N^\rho e_\rho^M e_\mu^{M'} \partial_{M'} \pi_M \\
&\equiv N (\bar{\nabla}_\mu \pi^\perp - \bar{K}_\mu^\rho \pi_\rho) + N^\rho (\bar{\nabla}_\mu \pi_\rho + \bar{K}_{\mu\rho} \pi^\perp), \quad (51)
\end{aligned}$$

where in the last step we used (A4). Plugging (49) and (51) into (50), simple manipulation yields

$$\begin{aligned}
& \eta_{MN} \partial_\mu \pi^M \partial_\nu \pi^N \\
&= \bar{g}^{\rho\sigma} (\bar{\nabla}_\mu \pi_\rho + \bar{K}_{\mu\rho} \pi^\perp) (\bar{\nabla}_\nu \pi_\sigma + \bar{K}_{\nu\sigma} \pi^\perp) \\
&\quad + (\bar{\nabla}_\mu \pi^\perp - \bar{K}_\mu^\rho \pi_\rho) (\bar{\nabla}_\nu \pi^\perp - \bar{K}_\nu^\sigma \pi_\sigma), \quad (52)
\end{aligned}$$

where the dependence on N and N_μ exactly gets cancelled. Putting all the above together, finally we write

$$\begin{aligned}
f_{\mu\nu} &= \bar{f}_{\mu\nu} - 2\bar{\nabla}_{(\mu} \pi_{\nu)} - 2\bar{K}_{\mu\nu} \pi^\perp \\
&\quad + (\bar{\nabla}_\mu \pi_\rho + \bar{K}_{\mu\rho} \pi^\perp) (\bar{\nabla}_\nu \pi^\rho + \bar{K}_\nu^\rho \pi^\perp) \\
&\quad + (\bar{\nabla}_\mu \pi^\perp - \bar{K}_\mu^\rho \pi_\rho) (\bar{\nabla}_\nu \pi^\perp - \bar{K}_\nu^\sigma \pi_\sigma). \quad (53)
\end{aligned}$$

(53) is the one of the main results in this work. Keep in mind that π^\perp is a function of π_μ given in (46).

Now our task is to plug the expression for π^\perp in terms of π^μ (46) into (53) in order to derive ‘‘covariant’’ expansion for $f_{\mu\nu}$ in terms of π^μ and 4-dimensional geometric quantities. We may write

$$f_{\mu\nu} - \bar{f}_{\mu\nu} = \sum_{n=1} f_{\mu\nu}^{(n)\text{Ebd}}, \quad (54)$$

where $f_{\mu\nu}^{(n)\text{Ebd}} \sim \mathcal{O}((\pi_\mu)^n)$ and superscript ‘‘Ebd’’ stands for ‘‘embedding’’. After some manipulations, we have, at the linear order,

$$f_{\mu\nu}^{(1)\text{Ebd}} = -2\bar{\nabla}_{(\mu} \pi_{\nu)}, \quad (55)$$

and at the quadratic order,

$$\begin{aligned}
f_{\mu\nu}^{(2)\text{Ebd}} &= \bar{\nabla}_\mu \pi_\rho \bar{\nabla}_\nu \pi^\rho - (\bar{K}_{\mu\nu} \bar{K}_{\rho\sigma} - \bar{K}_{\mu\rho} \bar{K}_{\nu\sigma}) \pi^\rho \pi^\sigma \\
&= \bar{\nabla}_\mu \pi_\rho \bar{\nabla}_\nu \pi^\rho - \bar{R}_{\mu\rho\nu\sigma} \pi^\rho \pi^\sigma, \quad (56)
\end{aligned}$$

where in the last step we used the Gauss relation

$$\bar{K}_{\mu\nu} \bar{K}_{\rho\sigma} - \bar{K}_{\mu\rho} \bar{K}_{\nu\sigma} = \bar{R}_{\mu\sigma\nu\rho}, \quad (57)$$

since the 5 dimensional bulk is flat. Note $f_{\mu\nu}^{(1)\text{Ebd}}$ in (55) and $f_{\mu\nu}^{(2)\text{Ebd}}$ in (56) exactly coincides with the results got in the RNC approach $f_{\mu\nu}^{(1)\text{RNC}}$ in (7) and $f_{\mu\nu}^{(2)\text{RNC}}$ in (8) respectively.

At the cubic order,

$$\begin{aligned}
f_{\mu\nu}^{(3)\text{Ebd}} &= \frac{1}{3} \pi^\alpha \pi^\beta \pi^\rho (\bar{K}_{\mu\nu} \bar{\nabla}_\alpha \bar{K}_{\beta\rho} - 3\bar{K}_{\alpha(\mu} \bar{\nabla}_{\nu)} \bar{K}_{\beta\rho}) \\
&\quad - \pi^\alpha \pi^\beta \bar{\nabla}_{(\mu} \pi_{\nu)} (2\bar{K}_{\nu)\alpha} \bar{K}_{\beta\rho} - \bar{K}_{\nu)\rho} \bar{K}_{\alpha\beta}). \quad (58)
\end{aligned}$$

At this point, one may be concerned about the presence of extrinsic curvature, since the final expressions should depend only on intrinsic 4 dimensional quantities. Fortunately, by employing the Gauss relation (57) many times, it is possible to recast (58) to be

$$f_{\mu\nu}^{(3)\text{Ebd}} = f_{\mu\nu}^{(3)\text{RNC}} - \frac{1}{3} \bar{\nabla}_{(\mu} (\pi^\alpha \pi^\beta \pi^\rho \bar{K}_{\nu)\rho} \bar{K}_{\alpha\beta}), \quad (59)$$

where $f_{\mu\nu}^{(3)\text{RNC}}$ is given in (9). Please note in deriving (59), we never use $\pi^\nu \bar{\nabla}_\mu \pi^\mu = 0$, which is the crucial assumption in the RNC approach (see Sec.II and [24] for details). The second term on the right-hand-side of (59), may be absorbed by a field redefinition

$$\pi_\mu \rightarrow \tilde{\pi}_\mu \equiv \pi_\mu + \Delta_\mu^{(3)} + \mathcal{O}((\pi_\mu)^4), \quad (60)$$

with

$$\Delta_\mu^{(3)} = \frac{1}{6} \pi^\alpha \pi^\beta \pi^\rho \bar{K}_{\mu\rho} \bar{K}_{\alpha\beta}. \quad (61)$$

This can be seen easily since $f_{\mu\nu}^{(1)\text{RNC}}(\Delta_\rho^{(3)})$ will exactly reproduce the second term on the right-hand-side of (59).

Similarly, at the quartic order, first we got the expression for $f_{\mu\nu}^{(4)\text{Ebd}}$ depending on the extrinsic curvature

$$\begin{aligned}
f_{\mu\nu}^{(4)\text{Ebd}} &= -\frac{1}{12} \pi^\alpha \pi^\beta \pi^\rho \pi^\sigma \left[3\bar{K}_{\alpha\beta} (\bar{K}_{\mu\nu} \bar{K}_{\rho\lambda} \bar{K}_\sigma^\lambda - \bar{K}_{\mu\lambda} \bar{K}_\nu^\lambda \bar{K}_{\rho\sigma}) \right. \\
&\quad \left. + \bar{K}_{\mu\nu} \bar{\nabla}_\alpha \bar{\nabla}_\beta \bar{K}_{\rho\sigma} - 3\bar{\nabla}_\mu \bar{K}_{\alpha\beta} \bar{\nabla}_\nu \bar{K}_{\rho\sigma} - 4\bar{K}_{\sigma(\mu} \bar{\nabla}_{\nu)} \bar{\nabla}_\alpha \bar{K}_{\beta\rho} \right] \\
&\quad - \frac{1}{3} \pi^\alpha \pi^\beta \pi^\rho \bar{\nabla}_{(\mu} \pi_{\nu)} (\bar{K}_{\nu)\sigma} \bar{\nabla}_\alpha \bar{K}_{\beta\rho} - 3\bar{\nabla}_{(\mu} \bar{K}_{\alpha\beta} \bar{K}_{\rho\sigma} - 3\bar{K}_{\nu)\alpha} \bar{\nabla}_\sigma \bar{K}_{\beta\rho}) \\
&\quad + \pi^\alpha \pi^\beta \bar{\nabla}_\mu \pi^\rho \bar{\nabla}_\nu \pi^\sigma \bar{K}_{\alpha\rho} \bar{K}_{\beta\sigma}. \quad (62)
\end{aligned}$$

Using the Gauss relation (57) again, after some manipulations, (62) can be recast as

$$f_{\mu\nu}^{(4)\text{Ebd}} = f_{\mu\nu}^{(4)\text{RNC}} + 2\bar{\nabla}_{(\mu}\pi^{\rho}\bar{\nabla}_{\nu)}\Delta_{\rho}^{(3)} - 2\bar{R}_{(\mu}{}^{\rho}{}_{\nu)}{}^{\sigma}\Delta_{\rho}^{(3)}\pi_{\sigma} - 2\bar{\nabla}_{(\mu}\Delta_{\nu)}^{(4)}, \quad (63)$$

where $f_{\mu\nu}^{(4)\text{RNC}}$ is given in (10), $\Delta_{\mu}^{(3)}$ is defined in (61), and

$$\Delta_{\mu}^{(4)} = -\frac{1}{24}\pi^{\alpha}\pi^{\beta}\pi^{\rho}\pi^{\sigma}(\bar{K}_{\rho\alpha}\bar{\nabla}_{\mu}\bar{K}_{\sigma\beta} + 2\bar{K}_{\alpha\mu}\bar{\nabla}_{\sigma}\bar{K}_{\rho\beta}). \quad (64)$$

It is interesting to note that although $f_{\mu\nu}^{(4)\text{Ebd}}$ itself does not coincide with $f_{\mu\nu}^{(4)\text{RNC}}$, their difference can be absorbed by the following field redefinition

$$\pi_{\mu} \rightarrow \tilde{\pi}_{\mu} = \pi_{\mu} + \Delta_{\mu}^{(3)} + \Delta_{\mu}^{(4)} + \mathcal{O}((\pi_{\mu})^5), \quad (65)$$

which is also consistent with (60). That is, we have

$$f_{\mu\nu}^{\text{Ebd}}(\pi_{\rho}) = f_{\mu\nu}^{\text{RNC}}(\tilde{\pi}_{\rho}), \quad (66)$$

where $\tilde{\pi}_{\mu}$ is given in (65) up to the fourth order.

V. CONCLUSION

The problem of covariant formulation for the Stückelberg analysis with a non-flat fiducial metric (or around a general background) has been known for some time. In [24] a covariant Stückelberg expansion was developed based on the Riemann normal coordinates. In this note we explore an alternative approach by considering the 4-dimensional curved space being a hypersurface embedded in a 5-dimensional Minkowski background, in which the Goldstone modes and the Stückelberg expansion can be performed in the standard manner. After eliminating one Goldstone modes through the constraint (26) and then projecting onto the 4-dimensional hypersurface, we are able to expand the Stückelbergized fiducial metric (11) in terms of 4-dimensional Goldstone modes π_{μ} as well as 4-dimensional geometric quantities, which are given in (55), (56), (59) and (63) respectively, up to the fourth order

in π_{μ} . Strikingly, the two approaches (RNC and embedding), although quite different from each other, give exactly coincide results after a simple field redefinition (65).

We expect the formalism developed in this note, may shed some light on the Stückelberg analysis as well as the decoupling limit of the massive gravity on a general background. There are some questions left to be answered. First it is important to find a geometric meaning for the field redefinition (65), which implies that the correct ‘‘covariant’’ Goldstone modes from the 4-dimensional point of view are actually nonlinear functions π_{μ} (instead of π_{μ} themselves). Moreover, as being emphasized before, higher codimensions may be needed in order to embed a general curved space into a flat background. It is thus interesting to generalize the formalism in this note to such a case.

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Appendix A: Projection of $\nabla_a A_b$

Let us consider a codimension-one hypersurface in an arbitrary background with metric g_{ab} . The normal vector n_a is normalized as $n_a n^a = +1$, which is spacelike. The induced metric on the hypersurface is $h_{ab} = g_{ab} - n_a n_b$. For an arbitrary vector field A_a , we may write

$$A_a \equiv n_a A^{\perp} + A_a^{\parallel}, \quad (A1)$$

with

$$A^{\perp} = n^a A_a, \quad A_a^{\parallel} = h_a^{a'} A_{a'}. \quad (A2)$$

It is easy to show that

$$h_a^{a'} h_b^{b'} \nabla_{a'} A_{b'} = D_a A_b^{\parallel} + K_{ab} A^{\perp}, \quad (A3)$$

$$h_a^{a'} n^{b'} \nabla_{a'} A_{b'} = D_a A^{\perp} - K_a^b A_b^{\parallel}, \quad (A4)$$

where D_a is the covariant derivative compatible with h_{ab} , $K_{ab} \equiv D_a n_b$ is the extrinsic curvature.

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